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Periodic Solutions of Abstract Differential Equations with Infinite Delay

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1. INTRODUCTION

The equation to be studied is

$$\frac{du}{dt} + Lu = \int_{-\infty}^{t} C(t, s)\, u(s)\, ds + f(u) + F(t), \tag{E}$$

where L and C are linear operators (at least L is unbounded, L is sectorial) in a real Banach space $\mathscr{B}$, f and F are quite smooth, $F(t+T) = F(t)$ and $C(t+T, s+T) = C(t, s)$ for some $T > 0$. The object is to give conditions to ensure that (E) has a T-periodic solution. When u is in a certain space, convergence properties for the integral will be required later.

The work proceeds as follows. First, (E) is written as a functional differential equation with a parameter λ, together with an associated homotopy h; if the homotopy has a fixed point for $\lambda = 1$, it is a periodic solution of (E). This is the content of Section 2.

In Section 3 the degree-theoretic work of Granas is summarized. This will enable us to show that if the homotopy h is compact and admissible, then the existence of an a *priori* bound on all possible T-periodic fixed points of h for $0 \leqslant \lambda \leqslant 1$ implies the existence of a T-periodic fixed point of h for $\lambda = 1$.

In Section 4 two nonlinear heat equations with memory are written as (E) and Liapunov function arguments yield a *priori* bounds on fixed points of the homotopy h. A similar Liapunov function also shows that h is compact through a Sobolev space argument using Rellich's lemma. Part of the novelty of this section is that the Liapunov functions need not have the standard upper and lower bounds; the a *priori* bound is derived mainly

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from the derivative of the Liapunov function. Moreover, although a Liapunov function is used on a functional differential equation, it does not utilize a Razumikhin technique. The argument does not yield a dissipative structure of the type discussed by Hale [10] in the search for periodic solutions as we show in Theorem 4.

2. A Homotopy

In this section we state the conditions needed to apply the theory of Granas. The examples of Section 4 show two ways in which these conditions can be realized.

It is supposed that the operator e^{-Lt} satisfies $|e^{-Lt}| \leqslant c_1 e^{-at}$ for $t > 0$ where c_1 and a are positive constants. Consider the companion to (E),

$$\frac{du}{dt} + Lu = \lambda \left[\int_{-\infty}^{t} C(t, s)\, u(s)\, ds + f(u) + F(t) \right] \tag{E$_\lambda$}$$

and the homotopy (on a space to be defined later)

$$h(\lambda, \varphi)(t) = \lambda \int_{-\infty}^{t} e^{-L(t-v)} \left[\int_{-\infty}^{v} C(v, s)\, \varphi(s)\, ds + f(\varphi(v)) + F(v) \right] dv.$$

It is also supposed that a convex subset Y of a Banach space can be found such that

$$\varphi \in Y \text{ implies } \varphi\colon (-\infty, \infty) \to \mathscr{B},\ \varphi(t+T) = \varphi(t),$$

for a certain closed subset X of Y,

$$h\colon [0, 1] \times X \to Y \text{ is a compact mapping, for } t > 0,\ h(\lambda, \varphi)(t) \in D(L),$$

and

$$\text{if } \varphi \text{ is a fixed point of } h \text{ then } \varphi \text{ satisfies } (\mathrm{E}_\lambda).$$

Obviously, these requirements induce strong conditions on C, f, and F. Briefly, they ask that $\varphi \in X$ implies that

$$\int_{-\infty}^{t} C(t, s)\, \varphi(s)\, ds + f(\varphi(t)) + F(t) \in \mathscr{B}$$

and is Hölder continuous in t for $t > 0$.

3. The Theory of Granas

The following definitions and results are from Granas [7]. Let Y be a convex subset of linear topological space, $A \subset X \subset Y$, and A closed in X. The space Y is at least Hausdorff.

Definition. (i) A continuous map $\varphi: X \to Y$ is *compact* if $\overline{\varphi(X)}$ is compact.

(ii) $h_\lambda := h(\lambda, \cdot): X \to Y$ is a *compact homotopy* if h is a homotopy and if for each $\lambda \in [0, 1]$, $h \mid \lambda \times X = h_\lambda$ is compact.

(iii) $\varphi: X \to Y$ is *admissible* with respect to A if φ is compact and $\varphi \mid A$ is fixed point free. Let $M_A(X, Y)$ denote the class of admissible maps with respect to A.

(iv) $\varphi \in M_A(X, Y)$ is *inessential* if there exists $\psi \in M_A(X, Y)$ such that $\varphi \mid A = \psi \mid A$ and ψ is without fixed points on X. Otherwise, $\varphi \in M_A(X, Y)$ is essential.

(v) A compact homotopy $h: [0, 1] \times X \to Y$ is *admissible* if for each $\lambda \in [0, 1]$, h_λ is admissible. Two mappings $\varphi, \psi \in M_A(X, Y)$ are homotopic in $M_A(X, Y)$, written $\varphi \sim \psi$, if there exists an admissible homotopy $h: [0, 1] \times X \to Y$ such that $h_0 = \psi$ and $h_1 = \varphi$.

(vi) F^* denotes the class of topological spaces which have the fixed point property for compact maps.

Theorem G1 (Granas). *Let Y be a connected space belonging to F^*, let $X \subset Y$ be closed, and let $A = \partial X$. If $\varphi: X \to Y$ is a constant map ($\varphi(x) = p$ for all $x \in X$) and $p \in X \setminus A$, then φ is essential.*

Theorem G2 (Granas). *Let $A = \bar{A} \subset X \subset Y$ and let $f \in M_A(X, Y)$ be an admissible function. Then the following are equivalent*:

(a) *$f: X \to Y$ is inessential.*

(b) $f \sim g$ in $M_A(X, Y)$ *where $g: X \to Y$ is without fixed points.*

The theory of Granas is applied in the following way:

(i) An a *priori* bound B_1 is found for all possible T-periodic solutions of (E_λ).

(ii) A convex subset Y of a Banach space of T-periodic functions is found and a set $X \subset \{\varphi \in Y: |\varphi| \leqslant B_1\}$ is constructed with the property that $h: X \to Y$ is a compact mapping.

(iii) In Theorem G1 we take $\varphi = h_0$ so that $h_0: X \to 0 \in X \backslash A$, where $A = \partial X$; thus, h_0 is essential.

(iv) By Theorem G2, taking $f = h_0$ and $g = h_1$, if h_1 is without fixed points, then h_0 is inessential, a contradiction.

Obviously, much care is needed in the construction of Y and X so that h is a compact mapping. Precise details are given in the proof of Theorem 1.

The theory of Granas has been applied to finite dimensional problems in [1, 2] to show that periodic solutions exist and in [5, 8, 9] to show that solutions of boundary value problems exist. Periodic solutions of infinite dimensional problems have been recently studied by DaPrato [3], DaPrato and Lunardi [4], Hale [10], and Lunardi [12] using different methods.

The survey book of Hale [10] describes in much detail research with dissipative systems. Periodic solutions are found when solutions are, essentially, uniformly bounded and uniformly ultimately bounded. By contrast, the Granas theory establishes the existence of periodic solutions, with the help of Liapunov functions, even when there are unbounded solutions (cf. [1] and our Theorem 4). In our examples, the Liapunov functions only operate on bounded solutions.

4. Periodic Solutions

For $\lambda \in [0, 1]$, consider the equation

$$u_t = u_{xx} + \lambda C(t, u(\cdot, x)), \tag{1}$$

where

$$u(t, 0) = u(t, 1) = 0, \tag{2}$$

$C(t, u(\cdot, x)) = C(t, u(s, x), -\infty < s \leqslant t)$ is a Volterra functional. We rewrite (1) as an abstract equation in $L^2(0, 1; R)$. Let $(A\varphi)(x) = -\partial^2\varphi(x)/\partial x^2$ for a smooth function φ on $[0, 1]$ and by using Friedrich's theorem, extend A to a self-adjoint, densely defined operator in $L^2(0, 1; R)$. Then $D(A) = H_0^1 \cap H^2$, where $H_0^1 = W_0^{1,2}(0, 1; R)$, $H^2 = W^{2,2}(0, 1; R)$. The norm on $H^j(0, 1)$ is denoted by $|\cdot|_{H^j}$. In particular, $|\cdot|_{H^0} = |\cdot|_{L^2}$.

The abstract version of (1) is

$$u'(t) + Au = \lambda C(t, u(\cdot)). \tag{3}$$

Note that in (3), we consider $C(t, u(\cdot))$ as a function $C: R \times L^2 \to L^2$ given by $C(t, u(\cdot))(x) = C(t, u(\cdot, x), -\infty < s \leqslant t)$.

We assume that $C(t+T, u(\cdot)) = C(t, u(\cdot))$ for some $T>0$. For $\varphi \in L^2(0, 1; R)$, define

$$\begin{gathered} \tilde{W} = \{\varphi \in C(R, H_0^1(0, 1)) \mid \varphi(t+T) = \varphi(t)\}, \\ |\varphi|_{\tilde{W}} = \sup\{|\varphi(t)|_{H^1} \mid 0 \leqslant t \leqslant T\}. \end{gathered} \tag{4}$$

The following result is probably known and easy to prove.

LEMMA 1. *The space $(\tilde{W}, |\cdot|_{\tilde{W}})$ defined by (4) is a Banach space.*

THEOREM 1. *Suppose the following conditions hold:*

(i) *there exists $B_1 > 0$ such that if $u(t)$ is a T-periodic solution of (3), then*

$$|u|_{\tilde{W}} < B_1;$$

(ii) *$C(\cdot, \varphi(\cdot))\colon R \to H^0$ is Hölder continuous whenever $\varphi \in (\tilde{W}, |\cdot|_{\tilde{W}})$ and is Hölder continuous;*

(iii) *$C\colon [0, T] \times \tilde{W} \to H^0$ takes bounded sets into bounded sets.*

(iv) *For any $\alpha > 0$, there exists $\beta > 0$ such that $[\varphi, \psi \in \tilde{W}, |\varphi|_{\tilde{W}} \leqslant \alpha, |\psi|_{\tilde{W}} \leqslant \alpha, t \geqslant 0]$ imply that*

$$|C(t, \varphi(\cdot)) - C(t, \psi(\cdot))|_{H^0} \leqslant \beta \sup_{t \in [0, T]} |\varphi(t) - \psi(t)|_{H^0}.$$

Then (3) has a T-periodic solution for $\lambda = 1$.

Proof. First, we set up the spaces for the Granas theory. By definition of the operator A, we refer to Henry [11, pp. 21, 26] and find $c > 0, a > 0$ with

$$\begin{aligned} &\text{(a)} \quad |e^{-At}| \leqslant ce^{-at}, \\ &\text{(b)} \quad |A^{1/2}e^{-At}| \leqslant ct^{-1/2}e^{-at}, \\ &\text{(c)} \quad |(e^{-At} - I)\,\varphi|_{H^0} \leqslant ct^{1/2}\,|A^{1/2}\varphi|_{H^0} \qquad \text{for all} \quad \varphi \in D(A^{1/2}). \end{aligned} \tag{5}$$

For the $B_1 > 0$ given in (i), by (iii) there exists a constant $C^* > 0$ such that

$$\sup_{t \in [0, T]} |C(t, \varphi(\cdot))|_{H^0} \leqslant C^* \qquad \text{whenever} \quad |\varphi|_{\tilde{W}} \leqslant B_1. \tag{6}$$

Let

$$L = c(T^{1/2} + cE)\,C^*,$$

where $E=\int_0^\infty t^{-1/2}e^{-at}\,dt$. Consider

$$Y=\{u\in \tilde{W} \mid |u(t_1)-u(t_2)|_{H^0}\leqslant L\,|t_1-t_2|^{1/2},\ t_1, t_2\in[0,T]\}$$

and (7)

$$X=\{u\in Y \mid |u|_{\tilde{W}}\leqslant B_1\}.$$

LEMMA 2. *If h: $R\times X\to Y$ is defined by*

$$h(\lambda,\varphi)(t,x)=\lambda\int_{-\infty}^{t} e^{-A(t-s)}C(s,\varphi(\cdot))\,ds$$

and if φ is a fixed point of h, then φ satisfies (3).

Proof. Let $J(t)=h(\lambda,\varphi)(t)$,

$$J(t)=J_0(t)+J_1(t),\qquad J_0(t)=\lambda\int_0^t e^{-A(t-s)}C(s,\varphi(\cdot))\,ds,$$

and $J_1(t)=\lambda\int_{-\infty}^0 e^{-A(t-s)}C(s,\varphi(\cdot))\,ds$. Since $C(t,\varphi(\cdot))$ is Hölder continuous, by Henry [11, p. 50]

$$(dJ_0(t)/dt)+AJ_0(t)=\lambda C(t,\varphi(\cdot)). \tag{8}$$

As

$$\int_{-\infty}^0 e^{As}C(s,\varphi(\cdot))\,ds \qquad \text{and} \qquad \int_{-\infty}^0 e^{-A(t-s)}C(s,\varphi(\cdot))\,ds$$

exist, by Friedman [6, p. 94, Lemma 1.2 and p. 101, Theorem 2.1], we have

$$\int_{-\infty}^0 e^{-A(t-s)}C(s,\varphi(\cdot))\,ds=e^{-At}\int_{-\infty}^0 e^{As}C(s,\varphi(\cdot))\,ds$$

and

$$(dJ_1(t)/dt)+AJ_1(t)=0. \tag{9}$$

By (8) and (9),

$$(dJ(t)/dt)+AJ(t)=\lambda C(t,\varphi(\cdot)).$$

Thus, if φ is a fixed point of h, then the right-hand side of h is differentiable and, hence, the left-hand side is also; this means that φ satisfies (3).

To see that $h(\lambda,\varphi)\in Y$, let $0\leqslant t_2<t_1\leqslant T$ and $\varphi\in X$; then

$$\begin{aligned}
&h(\lambda, \varphi)(t_1) - h(\lambda, \varphi)(t_2) \\
&\quad = \lambda \int_{-\infty}^{t_1} e^{-A(t_1 - s)} C(s, \varphi(\cdot))\, ds - \lambda \int_{-\infty}^{t_2} e^{-A(t_2 - s)} C(s, \varphi(\cdot))\, ds \\
&\quad = \lambda \int_{t_2}^{t_1} e^{-A(t_1 - s)} C(s, \varphi(\cdot))\, ds \\
&\qquad + [e^{-A(t_1 - t_2)} - I]\, \lambda \int_{-\infty}^{t_2} e^{-A(t_2 - s)} C(s, \varphi(\cdot))\, ds.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left| \lambda \int_{t_2}^{t_1} e^{-A(t_1 - s)} C(s, \varphi(\cdot))\, ds \right|_{H^0} \\
&\quad \leqslant \int_{t_2}^{t_1} c e^{-a(t_1 - s)} \, |C(s, \varphi(\cdot))|_{H^0}\, ds \\
&\quad \leqslant c\, |t_1 - t_2| \sup_{t \in [0, T]} |C(t, \varphi(\cdot))|_{H^0} \\
&\quad \leqslant c c^* T^{1/2}\, |t_1 - t_2|^{1/2}
\end{aligned}$$

by (6). Also

$$\begin{aligned}
&\left| (e^{-A(t_1 - t_2)} - I)\, \lambda \int_{-\infty}^{t_2} e^{-A(t_2 - s)} C(s, \varphi(\cdot))\, ds \right|_{H^0} \\
&\quad \leqslant c\, |t_1 - t_2|^{1/2} \left| A^{1/2} \int_{-\infty}^{t_2} e^{-A(t_2 - s)} C(s, \varphi(\cdot))\, ds \right|_{H^0} \\
&\quad \leqslant c\, |t_1 - t_2|^{1/2} \int_{-\infty}^{t_2} |A^{1/2} e^{-A(t_2 - s)}|\, |C(s, \varphi(\cdot))|_{H^0}\, ds \\
&\quad \leqslant c\, |t_1 - t_2|^{1/2} \int_{-\infty}^{t_2} c\, |t_2 - s|^{-1/2} e^{-a(t_2 - s)}\, |C(s, \varphi(\cdot))|_{H^0}\, ds \\
&\quad \leqslant c^2 E C^*\, |t_1 - t_2|^{1/2}
\end{aligned}$$

by (6). Thus,

$$|h(\lambda, \varphi)(t_1) - h(\lambda, \varphi)(t_2)|_{H^0} \leqslant L\, |t_1 - t_2|^{1/2}$$

and so $h(\lambda, \varphi) \in Y$.

LEMMA 3. *The homotopy $h(\lambda, \varphi)$ is compact.*

Proof. We first show that there is an $\alpha > 0$ such that

$$|J\varphi - J\psi|_{\tilde{W}} \leqslant \alpha \sup_{t \in [0,T]} |C(t, \varphi(\cdot)) - C(t, \psi(\cdot))|_{H^0}, \tag{10}$$

where $J\varphi = h(\lambda, \varphi)$ and $J\psi = h(\lambda, \psi)$. By Lemma 2 it follows that

$$(d/dt)(J\varphi - J\psi) + A(J\varphi - J\psi) = \lambda(C(t, \varphi(\cdot)) - C(t, \psi(\cdot))).$$

Define

$$V(t) = \int_0^1 [(J\varphi - J\psi)^2 + (J\varphi - J\psi)_x^2]\, dx$$

so that

$$V'(t) = 2\int_0^1 [(J\varphi - J\psi)(J\varphi - J\psi)_t + (J\varphi - J\psi)_x\,(J\varphi - J\psi)_{xt}]\, dx$$

(integration by parts)

$$= 2\int_0^1 [-(J\varphi - J\psi)_x^2 + \lambda(J\varphi - J\psi)(C(t, \varphi(\cdot)) - C(t, \psi(\cdot)))]\, dx$$

$$+ 2\int_0^1 [-(J\varphi - J\psi)_t^2 + \lambda(J\varphi - J\psi)_t\,(C(t, \varphi(\cdot)) - C(t, \psi(\cdot)))]\, dx.$$

There then exist $\mu, \gamma > 0$ such that

$$V'(t) \leqslant -\mu V(t) + \gamma \int_0^1 |C(t, \varphi(\cdot)) - C(t, \psi(\cdot))|^2\, dx.$$

Since $V(t)$ is T-periodic, there exists $\alpha > 0$ such that (10) holds. Incidentally, this shows that $h(\lambda, \varphi)$ is continuous in φ for fixed λ. Since $h(\lambda, \varphi)$ is uniformly continuous in λ for fixed φ, it follows that h is jointly continuous in (λ, φ).

Let $\{\varphi_n\}$ be a sequence in X; we will show that there is a subsequence $\{\varphi_{n_k}\}$ such that $\sup_{t \in [0,T]} |\varphi_{n_k}(t) - \varphi_{n_m}(t)|_{H^0} \to 0$ as $k, m \to \infty$. This means that $\{J\varphi_{n_k}\}$ is a Cauchy sequence in $\tilde{W}$, a Banach space, and so h is a compact mapping.

Let $\{t_n\}$ be the sequence of all rationals in $[0, T]$. Since $\{\varphi_n(t_1)\}$ is bounded in $X \subset H^1(0, 1)$, by Rellich's lemma, there is a subsequence $\{\varphi_{n,1}(t_1)\}$ converging in $H^0(0, 1)$. Next, the sequence $\{\varphi_{n,1}(t_2)\}$ has a convergent subsequence $\{\varphi_{n,2}(t_2)\}$ converging in $H^0(0, 1)$. In the kth step we extract a convergent subsequence $\{\varphi_{n,k}(t_k)\}$. Write $\psi_n = \varphi_{n,n}$. Then $\{\psi_n(t_k)\}$ converges in $H^0(0, 1)$ for every k. Since $\{\psi_n\} \subset X$, $\{\psi_n\}$ is equicontinuous: for each $\varepsilon > 0$ there exists $\delta > 0$ ($\varepsilon = 2\delta^{1/2}L$) such that $|s_1 - s_2| < \delta$ implies that $|\psi_n(s_1) - \psi_n(s_2)|_{H^0} < \varepsilon$. Since $[0, T]$ is compact, there exist a finite number of points t_k, say $t_1, t_2, ..., t_N$, such that the intervals $B_k = \{t \mid |t - t_k| < \delta/2\}$, $k = 1, 2, ..., N$, cover $[0, T]$. Then for each $t \in [0, T]$ there exists t_k with $t \in B_k$. Thus,

$$\begin{aligned} |\psi_n(t) - \psi_m(t)|_{H^0} &\leqslant |\psi_n(t) - \psi_n(t_k)|_{H^0} \\ &\quad + |\psi_n(t_k) - \psi_m(t_k)|_{H^0} + |\psi_m(t_k) - \psi_m(t)|_{H^0} \\ &\leqslant 2\varepsilon + |\psi_n(t_k) - \psi_m(t_k)|_{H^0}. \end{aligned}$$

For each k, $1 \leqslant k \leqslant N$, there is a positive integer n_k with

$$|\psi_n(t_k) - \psi_m(t_k)|_{H^0} < \varepsilon \qquad \text{if} \quad m, n \geqslant n_k.$$

Using this in the above result, we get

$$\sup_{t \in [0, T]} |\psi_n(t) - \psi_m(t)|_{H^0} < 3\varepsilon \qquad \text{if} \quad m, n \geqslant \max\{n_1, ..., n_k\} \stackrel{\text{def}}{=} Q.$$

Thus,

$$\sup_{t \in [0, T]} |\psi_n(t) - \psi_m(t)|_{H^0} < 3\varepsilon \qquad \text{if} \quad m, n \geqslant Q.$$

This means that

$$|J\psi_n - J\psi_m|_{\tilde{W}} \leqslant 3\varepsilon\beta(\gamma/\mu)^{1/2},$$

where β is defined in (iv) for $\alpha = B_1$, so $\{J\psi_n\}$ is a Cauchy sequence in $\tilde{W}$. Also, Y is closed in $\tilde{W}$ and so the limit function is in Y. This proves Lemma 3.

Now we are ready to complete the proof of Theorem 1. The sets X and Y are defined by (7). Define

$$\tilde{A} = \{\varphi \in Y \mid |\varphi|_{\tilde{W}} = B_1\}.$$

Now Y is a convex subset of a Banach space which is an F^* space. Also, X is closed in Y and for $\tilde{A}$ defined above, $\tilde{A} = \partial X$ in the topological space Y.

Referring to Granas' Theorem G1, we take $\varphi = h_0$ so that $h_0: X \to 0 \in X \backslash \tilde{A}$ implies that h_0 is essential. Thus, in Granas' Theorem G2 we take $f = h_0$ and $g = h_1$. If h_1 is without fixed points, then h_0 is inessential, a contradiction. Hence, h_1 has a fixed point in Y which satisfies (3) by Lemma 2. This completes the proof.

EXAMPLE 1. Consider the equation

$$u_t = u_{xx} + \lambda \left[f(u) + F(t) + \int_{-\infty}^{t} B(t, s)\, u_t(s, x)\, ds \right], \tag{11}$$

where

$$u(t, 0) = u(t, 1) = 0; \tag{12}$$

$$\begin{gathered} uf(u) \leqslant ku^2 + M \text{ for some } k \in (0, \pi^2) \text{ and } M > 0, \\ F(t+T) = F(t),\ B(t+T, s+T) = B(t, s) \text{ for some } T > 0, \\ f \in C^1;\ B \text{ continuous;} \end{gathered} \tag{13}$$

$$\begin{array}{ll} \text{(a)} & B(t, t) \text{ is differentiable on } [0, T], \\ \text{(b)} & \int_{-\infty}^{0} \tilde{B}(v)\, dv < 1, \text{ where } \tilde{B}(v) = \sup_{t \in [0, T]} \{ |B(t, t+v)| \}, \\ \text{(c)} & \int_{-\infty}^{0} |B_v(t, t+v)|\, dv \leqslant D_1 \text{ for some } D_1 > 0, \text{ all } t \in [0, T], \\ \text{(d)} & |B_v(t, t+v) - B_v(s, s+v)| \leqslant |t-s|^{\theta}\, b(v) \text{ for } t, s \in [0, T] \\ & \text{where } \theta \in (0, 1] \text{ and } b(t) \in L^1(-\infty, 0]; \end{array} \tag{14}$$

$$\begin{array}{ll} \text{(a)} & \text{for each } D > 0 \text{ there is a } K > 0 \text{ such that } |t|, |s| \leqslant D \text{ imply} \\ & \text{that } |f(t) - f(s)| \leqslant K\, |t-s| \text{ (this is not independent} \\ & \text{of (13));} \\ \text{(b)} & \text{there is a } K_1 > 0 \text{ such that } |F(t) - F(s)| \leqslant K_1\, |t-s|. \end{array} \tag{15}$$

We rewrite (11) as an abstract equation

$$u_t + Au = \lambda C(t, u(\cdot)), \tag{16}$$

where

$$C(t, u(s, x), -\infty < s \leqslant t) = f(u(t, x)) + F(t) + B(t, t)\, u(t, x) - \int_{-\infty}^{0} B_v(t, t+v)\, u(t+v, x)\, dv \tag{17}$$

for a bounded function u.

LEMMA 4. *Suppose that* (13), (14)(b) *hold. Then there is a constant* $B_1 > 0$ *such that if* $u(t)$ *is a* T*-periodic solution of* (16), *then* $|u|_{\tilde{W}} < B_1$.

(Note that we are dealing with an element of the equivalence class of L^2 in the proof of Lemma 4.)

Proof. Let $u(t, x)$ be a T-periodic solution of (11) and define

$$V_1(t) = \int_0^1 u^2(t, x)\, dx$$

so that

$$V_1'(t) = 2\int_0^1 u(t,x)\, u_t(t,x)\, dx$$

$$= 2\int_0^1 u(t,x)\left[u_{xx} + \lambda f(u) + \lambda F(t) + \lambda \int_{-\infty}^t B(t,s)\, u_t(s)\, ds\right] dx.$$

Integration by parts and use of (12) yield $\int_0^1 u u_{xx}\, dx = -\int_0^1 u_x^2\, dx$ so that by (13) we have

$$V_1'(t) \leqslant -2\int_0^1 u_x^2\, dx + 2\lambda k \int_0^1 u^2 dx + 2\lambda M$$

$$+ 2\lambda \int_0^1 |u|\, |F(t)|\, dx + \int_0^1 2\lambda\, |u| \int_{-\infty}^t |B(t,s)|\, |u_t(s,x)|\, ds\, dx$$

Now, find $\delta > 0$ so small that $k + \delta^2\pi^2 < \pi^2$. We then have

$$V_1'(t) \leqslant \int_0^1 \Big[-2u_x^2 + 2\lambda k u^2 + 2\lambda M + 2\lambda\, |u|\, |F(t)|$$

$$+ \lambda\delta^2 u^2 \int_{-\infty}^t |B(t,s)|\, ds + (1/\delta^2) \int_{-\infty}^t |B(t,s)|\, u_t^2(s,x)\, ds \Big] dx.$$

Using the facts that $|F(t)|$ is bounded, $k < \pi^2$, $\int_0^1 \pi^2 u^2\, dx \leqslant \int_0^1 u_x^2\, dx$, $\int_{-\infty}^0 \tilde{B}(v)\, dv < 1$, find positive constants $\alpha_1, \beta_1, \gamma_1$ with

$$\text{(i)}\quad V_1'(t) \leqslant \lambda\gamma_1 - \alpha_1 \int_0^1 (u^2 + u_x^2)\, dx$$

$$+ \lambda\beta_1 \int_0^1 \int_{-\infty}^t |B(t,s)|\, u_t^2(s,x)\, ds\, dx.$$

Next, define

$$V_2(t) = \int_0^1 \left[(1/2)\, u_x^2 - \lambda \int_0^u f(\xi)\, d\xi\right] dx$$

and obtain

$$V_2'(t) = \int_0^1 \{u_x u_{xt} - \lambda f(u)\, u_t\}\, dx.$$

From (12) we have $u_t(t, 0) = u_t(t, 1) = 0$ so that

$$\int_0^1 u_x u_{xt}\, dx = \int_0^1 u_x u_{tx}\, dx = u_x u_t \Big|_0^1 - \int_0^1 u_t u_{xx}\, dx$$
$$= -\int_0^1 u_t u_{xx}\, dx = -\int_0^1 u_t \Big[u_t - \lambda f(u) - \lambda F(t)$$
$$- \lambda \int_{-\infty}^t B(t, s)\, u_t(s, x)\, ds \Big] dx$$

and hence

$$\text{(ii)} \quad V_2'(t) = \int_0^1 \left\{ -u_t^2 + \lambda u_t \int_{-\infty}^t B(t, s)\, u_t(s, x)\, ds + \lambda u_t F(t) \right\} dx$$
$$\leqslant -\left(1 - \frac{1}{2}\int_{-\infty}^t |B(t, s)|\, ds\right) \int_0^1 u_t^2\, dx + \lambda \int_0^1 |u_t|\, |F(t)|\, dx$$
$$+ \frac{1}{2}\int_0^1 \int_{-\infty}^t |B(t, s)|\, u_t^2(s, x)\, ds\, dx.$$

As u and V_2 are T-periodic we have

$$0 = V_2(T) - V_2(0)$$
$$\leqslant -\left(1 - \frac{1}{2}\right) \int_0^T \int_0^1 u_t^2\, dx\, dt + \int_0^T \int_0^1 |u_t|\, |F(t)|\, dx\, dt$$
$$+ \frac{1}{2}\int_0^T \int_0^1 \int_{-\infty}^0 |B(t, t+s)|\, |u_t(s+t)|^2\, ds\, dx\, dt$$
$$\leqslant -\frac{1}{2}\int_0^T \int_0^1 u_t^2\, dx\, dt + \int_0^T \int_0^1 |u_t|\, |F(t)|\, dx\, dt$$
$$+ \frac{1}{2}\int_0^1 \int_{-\infty}^0 |\tilde{B}(s)| \int_0^T u_t^2(t+s)\, dt\, ds\, dx$$

(because u is T-periodic)

$$= -\frac{1}{2}\left[1 - \int_{-\infty}^0 \tilde{B}(s)\, ds\right] \int_0^T \int_0^1 u_t^2\, dx\, dt + \int_0^T \int_0^1 |u_t|\, |F(t)|\, dx\, dt.$$

By (14)(b) and the fact that $|F(t)|$ is bounded, there is a $\beta_2 > 0$ with

$$\int_0^T \int_0^1 u_t^2\, dx\, dt \leqslant \beta_2.$$

In the same way we consider V_1 and find that

$$\begin{aligned}0 &= V_1(T) - V_1(0)\\ &\leqslant -\alpha_1 \int_0^T \int_0^1 (u^2 + u_x^2)\, dx\, dt + \lambda\beta_1 \int_{-\infty}^0 \tilde{B}(s)\, ds \int_0^T \int_0^1 u_t^2\, dx\, dt\\ &\quad + \lambda\gamma_1 T.\end{aligned}$$

Thus,

$$\alpha_1 \int_0^T \int_0^1 (u^2 + u_x^2)\, dx\, dt \leqslant \beta_1 \beta_2 + \lambda\gamma_1 T.$$

Hence, there is an $\bar{M} > 0$ with

$$\text{(iii)} \quad \int_0^T \int_0^1 (u^2 + u_x^2 + u_t^2)\, dx\, dt \leqslant \bar{M}.$$

Now, let $V(t) = V_1(t) + V_2(t)$ and use (i) and (ii) to obtain

$$\begin{aligned}V'(t) \leqslant &-\alpha \int_0^1 (u^2 + u_x^2 + u_t^2)\, dx\\ &+ \beta \int_0^1 \int_{-\infty}^t |B(t, s)|\, u_t^2(s, x)\, ds\, dx + \gamma\end{aligned}$$

for some positive constants α, β, γ. Now u and u_x are continuous, so by (iii) there is a $t_1 \in [0, T]$ with

$$T \int_0^1 (u^2(t_1, x) + u_x^2(t_1, x))\, dx = \int_0^T \int_0^1 (u^2 + u_x^2)\, dx\, dt \leqslant \bar{M}.$$

Thus, by Sobolev's inequality

$$\text{(iv)} \quad \sup_{0 \leqslant x \leqslant 1} |u(t_1, x)| \leqslant \int_0^1 (|u(t_1, x)| + |u_x(t_1, x)|)\, dx \leqslant 2 + (\bar{M}/T)$$

since $|u| + |u_x| \leqslant 2 + u^2 + u_x^2$. We write this as

$$\text{(v)} \quad \sup_{0 \leqslant x \leqslant 1} |u(t_1, x)| \leqslant 2 + (\bar{M}/T) \stackrel{\text{def}}{=} c_1.$$

This means that

$$\text{(vi)}\quad \left|\int_0^{u(t_1,x)} f(\xi)\,d\xi\right| \leqslant \max_{0\leqslant |u|\leqslant c_1}\left|\int_0^u f(\xi)\,d\xi\right| \overset{\text{def}}{=} c_2$$

and

$$\text{(vii)}\quad V(t_1)\leqslant \int_0^1 \left(u^2(t_1,x)+\frac{1}{2}u_x^2(t_1,x)\right)dx+c_2.$$

Hence, if $t_1\leqslant t\leqslant t_1+T$ we have

$$\begin{aligned}
V(t) &= V(t_1)+\int_{t_1}^t V'(s)\,ds\\
&\leqslant V(t_1)+\beta\int_{t_1}^{t_1+T}\int_0^1\int_{-\infty}^0 |B(v,v+s)|\,u_t^2(v+s)\,ds\,dx\,dv+\gamma T\\
&= V(t_1)+\beta\int_0^T\int_0^1\int_{-\infty}^0 |B(v,v+s)|\,u_t^2(v+s)\,ds\,dx\,dv+\gamma T\\
&\leqslant V(t_1)+\beta\int_{-\infty}^0 \tilde{B}(s)\,ds\int_0^1\int_0^T u_t^2(t,x)\,dt\,dx+\gamma T\\
&\leqslant V(t_1)+\beta\bar{M}+\gamma T
\end{aligned}$$

by (iii) and (14)(b). But $V(t_1)\leqslant c_2+\bar{M}/T$ and so

$$V(t)\leqslant c_2+\beta\bar{M}+\gamma T+\bar{M}/T \overset{\text{def}}{=} c_3.$$

From (13) we have

$$\text{(viii)}\quad uf(u)\leqslant ku^2+M,\qquad k\in(0,\pi^2).$$

Define

$$c^* = \sup_{|u|\leqslant 1}\left|\int_0^u f(s)\,ds\right|$$

and use $V(t)\leqslant c_3$ to write

$$\text{(ix)}\quad \int_0^1\left[u^2+\frac{1}{2}u_x^2\right]dx\leqslant c_3+\lambda\int_0^1\int_0^u f(s)\,ds\,dx.$$

Case 1. If $|u|\leqslant 1$, then

$$\int_0^u f(\xi)\,d\xi\leqslant c^*.$$

Case 2. Suppose $|u| > 1$.

(a) If $u(t, x) > 1$, then

$$\begin{aligned}\int_0^u f(s)\,ds &= \int_0^1 f(s)\,ds + \int_1^u f(s)\,ds \\ &\leqslant c^* + \int_1^u [sf(s)/s]\,ds \qquad \text{(using (viii))} \\ &\leqslant c^* + \int_1^u (ks + (M/s))\,ds \\ &\leqslant c^* + \frac{1}{2}(k+1)\,u^2 + M^*, \qquad \text{some } M^* > 0.\end{aligned}$$

(b) If $u(t, x) < -1$, then

$$\begin{aligned}\int_0^u f(s)\,ds &= -\int_0^{-u} f(-s)\,ds = -\int_0^{|u|} f(-s)\,ds \\ &= -\int_0^1 f(-s)\,ds - \int_1^{|u|} f(-s)\,ds \\ &\leqslant c^* + \int_1^{|u|} [-sf(-s)/s\,ds \\ &\leqslant c^* + \int_1^{|u|} (ks + (M/s))\,ds \\ &\leqslant c^* + \frac{1}{2}(k+1)\,u^2 + M^*, \qquad \text{as before.}\end{aligned}$$

Thus,

$$\lambda \int_0^u f(s)\,ds \leqslant c^* + \frac{1}{2}(k+1)\,u^2 + M^*$$

and

$$\lambda \int_0^1 \int_0^u f(s)\,ds\,dx \leqslant c^* + \frac{1}{2}(k+1) \int_0^1 u^2\,dx + M^*.$$

From (ix) we have

$$\int_0^1 \left[u^2 + \frac{1}{2}u_x^2\right] dx \leqslant c_3 + c^* + \frac{1}{2}(k+1) \int_0^1 u^2\,dx + M^*.$$

This yields

$$\frac{1}{2}\int_0^1 [u^2+(1-(k/\pi^2))\,u_x^2]\,dx \leqslant c_3+c^*+M^*.$$

Hence, there is a $B_1>0$ with

$$\sup_{t\in[0,T]}\int_0^1 [u^2+u_x^2]\,dx \leqslant B_1.$$

This completes the proof.

We remark that when B is of convolution type then (14)(d) is not needed in the next result.

LEMMA 5. *Suppose that* (13)–(15) *hold. Then* $C(t,\varphi(\cdot))$ *given in* (17) *is Hölder continuous whenever* $\varphi\in(\tilde{W},|\cdot|_{\tilde{W}})$ *with* φ *Hölder continuous.*

Proof. Let $\varphi\in(\tilde{W},|\cdot|_{\tilde{W}})$ be Hölder continuous. Then there is a $k\in(0,1]$, $L_1>0$ such that

$$|\varphi(t_1)-\varphi(t_2)|_{H^0}\leqslant L_1\,|t_1-t_2|^k.$$

For $t_1, t_2\in[0,T]$, we have

$$\begin{aligned}
&|B(t_1,t_1)\,\varphi(t_1)-B(t_2,t_2)\,\varphi(t_2)|_{H^0}\\
&\quad\leqslant |B(t_1,t_1)\,\varphi(t_1)-B(t_1,t_1)\,\varphi(t_2)|_{H^0}\\
&\qquad+|B(t_1,t_1)\,\varphi(t_2)-B(t_2,t_2)\,\varphi(t_2)|_{H^0}\\
&\quad\leqslant |B(t_1,t_1)|\;|\varphi(t_1)-\varphi(t_2)|_{H^0}\\
&\qquad+|B(t_1,t_1)-B(t_2,t_2)|\;|\varphi(t_2)|_{H^0}.
\end{aligned}$$

From (14) we have

$$\begin{gathered}
|B(t,t)|\leqslant B^* \text{ for some } B^*>0, \text{ all } t\in[0,T],\\
|B(t,t)-B(s,s)|\leqslant L_B\,|t-s| \text{ for some } L_B>0 \text{ and all } t,s\in[0,T].
\end{gathered}\tag{18}$$

Thus

$$\begin{aligned}
&|B(t_1,t_1)\,\varphi(t_1)-B(t_2,t_2)\,\varphi(t_2)|_{H^0}\\
&\quad\leqslant B^*L_1\,|t_1-t_2|^k+L_B\,|t_1-t_2|\;|\varphi|_{\tilde{W}}.
\end{aligned}\tag{19}$$

Next,

$$
\begin{aligned}
&\left|\int_{-\infty}^{0} B_v(t_1, t_1+v)\, \varphi(t_1+v)\, dv - \int_{-\infty}^{0} B_v(t_2, t_2+v)\, \varphi(t_2+v)\, dv\right|_{H^0} \\
&\quad \leqslant \left|\int_{-\infty}^{0} B_v(t_1, t_1+v)\, \varphi(t_1+v)\, dv \right. \\
&\qquad \left. - \int_{-\infty}^{0} B_v(t_1, t_1+v)\, \varphi(t_2+v)\, dv\right|_{H^0} \\
&\qquad + \left|\int_{-\infty}^{0} B_v(t_1, t_1+v)\, \varphi(t_2+v)\, dv \right. \\
&\qquad \left. - \int_{-\infty}^{0} B_v(t_2, t_2+v)\, \varphi(t_2+v)\, dv\right|_{H^0} \\
&\quad \leqslant \int_{-\infty}^{0} |B_v(t_1, t_1+v)|\, |\varphi(t_2+v) - \varphi(t_1+v)|_{H^0}\, dv \\
&\qquad + \int_{-\infty}^{0} |B_v(t_1, t_1+v) - B_v(t_2, t_2+v)|\, |\varphi(t_2+v)|_{H^0}\, dv \\
&\quad \leqslant L_1 D_1\, |t_1 - t_2|^k + |\varphi|_{\tilde{W}} \int_{-\infty}^{0} b(v)\, dv\, |t_1 - t_2|^{\theta}.
\end{aligned} \tag{20}
$$

Note that

$$
\begin{aligned}
&|f(\varphi(t_1)) - f(\varphi(t_2))|_{H^0} + |F(t_1) - F(t_2)|_{H^0} \\
&\qquad \leqslant K L_1\, |t_1 - t_2|^k + K_1\, |t_1 - t_2|.
\end{aligned} \tag{21}
$$

Combining (19), (20), (21), there is an $\alpha \in (0, 1]$ and a constant k_2 depending on $|\varphi|_{\tilde{W}}$ such that

$$
|C(t_1, \varphi(\cdot)) - C(t_2, \varphi(\cdot))|_{H^0} \leqslant k_2\, |t_1 - t_2|^{\alpha}.
$$

This proves Lemma 5.

LEMMA 6. *Suppose* (13)–(15) *hold. Then* $C\colon [0, T] \times \tilde{W} \to H^0$ *takes bounded sets into bounded sets and condition* (iv) *of Theorem* 1 *holds.*

Proof. By (15), for any $\varphi \in (\tilde{W}, |\cdot|_{\tilde{W}})$, there is a $K > 0$ depending on $|\varphi|_{\tilde{W}}$ such that

$$
|f(\varphi)|_{H^0} \leqslant K\, |\varphi|_{H^0} + |f(0)|.
$$

Let $\|F\| = \sup\{|F(t)| : 0 \leqslant t \leqslant T\}$, then

$$
|C(t, \varphi(\cdot))|_{H^0} \leqslant (K + B^* + D_1)\, |\varphi|_{H^0} + |f(0)| + \|F\|
$$

for all $t \in [0, T]$.

For $\varphi, \psi \in X$ we have

$$|C(t, \varphi(\cdot)) - C(t, \psi(\cdot))|_{H^0} \leqslant (K + B^* + D_1) \sup_{t \in [0, T]} |\psi(t) - \varphi(t)|_{H^0}.$$

Thus (iv) of Theorem 1 holds. This completes the proof.

THEOREM 2. *Suppose that* (13), (14), (15) *hold. Then* (16) *has a T-periodic solution for* $\lambda = 1$.

We note that by Lemmas 4, 5, 6 all conditions of Theorem 1 are satisfied for C defined by (17).

EXAMPLE 2. We now consider the scalar equation

$$u_t = u_{xx} + \lambda\left[f(u) + F(t) + \int_{-\infty}^{t} B(t, s)\, u_{xx}(s, x)\, ds\right], \tag{22}$$

where

$$u(t, 0) = u(t, 1) = 0, \tag{23}$$

$$\begin{gathered} F(t),\ B(t, s) \text{ are continuous and satisfy} \\ F(t+T) = F(t),\ B(t+T, s+T) = B(t, s) \text{ for some } T > 0, \end{gathered} \tag{24}$$

$$\begin{aligned}
&\text{(a)}\quad B(t, t) \text{ is differentiable on } [0, T],\\
&\text{(b)}\quad \tilde{B}(s) \in L^1(-\infty, 0] \text{ where } \tilde{B}(s) = \sup_{t \in [0, T]} \{|B(t, t+s)|\},\\
&\text{(c)}\quad \int_{-\infty}^{t} |B_t(t, s)|\, ds \leqslant D_2 \text{ for some } D_2 > 0 \text{ and all } t \in [0, T], \qquad (25)\\
&\text{(d)}\quad |B(t_1, s) - B(t_2, s)| \leqslant b(s)\, |t_1 - t_2|^{\theta} \text{ for some } \theta \in (0, 1],\\
&\qquad\ b \in L^1(-\infty, T], \text{ and for all } t_1, t_2 \in [0, T],\\
&\text{(e)}\quad |(\partial/\partial_1) B(t_1, t_1 + v) - (\partial/\partial_1) B(t_2, t_2 + v)| \leqslant b^*(v)\, |t_1 - t_2|^{\alpha}\\
&\qquad\ \text{for all } t_1, t_2 \in [0, T],
\end{aligned}$$

for some $\alpha \in (0, 1]$ and $b^*(v) \in L^1(-\infty, 0]$, where $(\partial/\partial_1) B(t, s)$ denotes the derivative of B with respect to the first argument,

$$\begin{gathered} \text{there is a continuous function } \varphi\colon R \to R^+ \text{ with} \\ \left|\int_0^{\pm\infty} \varphi(u)\, du\right| < \infty \text{ and there is a } k \in R^+ \text{ with } f'(u) \leqslant k + \varphi(u), \end{gathered} \tag{26}$$

$$\text{for the } k \text{ in (26) we have } (k/\pi^2) + \int_{-\infty}^{0} \tilde{B}(v)\, dv < 1. \tag{27}$$

Remark. Note that if we integrate $f'(u) \leqslant k + \varphi(u)$ we obtain $uf(u) \leqslant ku^2 + M\,|u|$ for some constant $M > 0$.

LEMMA 7. *If* (24), (25), (26), (27) *hold, then there is a* $B_1 > 0$ *such that if* $u(t, x)$ *is a T-periodic solution of* (22), *then* $|u|_{\bar{W}} < B_1$.

Proof. Let $u(t, x)$ be a T-periodic solution of (22) and define

$$V_1(t) = \int_0^1 u^2(t, x)\,dx$$

so that

$$\begin{aligned} V_1'(t) &= 2\int_0^1 uu_t\,dx \\ &= 2\int_0^1 u\left[u_{xx} + \lambda f(u) + \lambda F(t) + \lambda\int_{-\infty}^{t} B(t, s)\,u_{xx}(s, x)\,ds\right]dx. \end{aligned}$$

Note that $\int_0^1 uu_{xx}\,dx = -\int_0^1 u_x^2\,dx$ so that by setting $t - s = -v$ and using $uf(u) \leqslant ku^2 + M$ we obtain

$$\begin{aligned} V_1'(t) \leqslant &-2\int_0^1 u_x^2\,dx + 2k\int_0^1 u^2\,dx \\ &+ 2M\int_0^1 |u|\,dx + 2\int_0^1 |u|\,|F(t)|\,dx \\ &+ 2\lambda\int_{-\infty}^0 B(t, t+s)\int_0^1 u(t, x)\,u_{xx}(t+s, x)\,dx\,ds. \end{aligned}$$

Integration of the last term by parts yields

$$\begin{aligned} V_1'(t) \leqslant &-2\int_0^1 u_x^2\,dx + 2k\int_0^1 u^2\,dx \\ &+ 2M\int_0^1 |u|\,dx + 2\int_0^1 |u|\,|F(t)|\,dx \\ &- 2\lambda\int_{-\infty}^0 B(t, t+s)\int_0^1 u_x(t, x)\,u_x(t+s, x)\,dx\,ds \end{aligned}$$

and so

$$\begin{aligned}
V_1'(t) \leqslant & -2\int_0^1 u_x^2\,dx + 2k\int_0^1 u^2\,dx + 2M\int_0^1 |u|\,dx \\
& + 2\int_0^1 |u|\,|F(t)|\,dx + \int_{-\infty}^0 |B(t,t+s)|\,ds \int_0^1 u_x^2\,dx \\
& + \int_{-\infty}^0 |B(t,t+s)| \int_0^1 u_x^2(t+s,x)\,dx\,ds \\
\leqslant & -2\int_0^1 u_x^2\,dx + 2k\int_0^1 u^2\,dx + 2M\int_0^1 |u|\,dx \\
& + 2\int_0^1 |u|\,|F(t)|\,dx + \int_{-\infty}^0 \tilde{B}(v)\,dv \int_0^1 u_x^2(t,x)\,dx \\
& + \int_{-\infty}^0 \tilde{B}(v) \int_0^1 u_x^2(t+v,x)\,dx\,dv.
\end{aligned}$$

Hence,

$$\begin{aligned}
V_1(T) - V_1(0) \leqslant & -2\int_0^T\int_0^1 u_x^2\,dx\,dt + 2k\int_0^T\int_0^1 u^2\,dx\,dt \\
& + 2M\int_0^T\int_0^1 |u|\,dx\,dt + 2\int_0^T\int_0^1 |u|\,|F(t)|\,dx\,dt \\
& + \int_{-\infty}^0 \tilde{B}(v)\,dv \int_0^T\int_0^1 u_x^2(t,x)\,dx\,dt \\
& + \int_{-\infty}^0 \tilde{B}(v) \int_0^T\int_0^1 u_x^2(t+v,x)\,dx\,dv \\
\leqslant & -2\left(1 - \int_{-\infty}^0 \tilde{B}(v)\,dv\right)\int_0^T\int_0^1 u_x^2\,dx\,dt \\
& + (2k/\pi^2)\int_0^T\int_0^1 u_x^2\,dx\,dt \\
& + 2M\int_0^T\int_0^1 |u|\,dx\,dt + 2\int_0^T\int_0^1 |u|\,|F(t)|\,dx\,dt,
\end{aligned}$$

where we have used $\pi^2\int_0^1 u^2\,dx \leqslant \int_0^1 u_x^2\,dx$. By this and (27), there is a $c_1 > 0$ with

$$\int_0^T\int_0^1 u_x^2\,dx\,dt \leqslant c_1. \tag{28}$$

Next, define

$$V_2(t) = \int_0^1 u_x^2(t, x)\,dx$$

so that

$$\begin{aligned}
V_2'(t) &= 2\int_0^1 u_x u_{xt}\,dx = 2u_t u_x\Big|_0^1 - 2\int_0^1 u_t u_{xx}\,dx \\
&= -2\int_0^1 u_{xx}\Big[u_{xx} + \lambda f(u) + \lambda F(t) \\
&\quad + \lambda \int_{-\infty}^t B(t, s)\,u_{xx}(s, x)\,ds\Big]\,dx \\
&= -2\int_0^1 u_{xx}^2\,dx - 2\lambda \int_0^1 f(u)\,u_{xx}\,dx - 2\lambda \int_0^1 u_{xx} F(t)\,dx \\
&\quad - 2\lambda \int_0^1 u_{xx}(t, x) \int_{-\infty}^0 B(t, t+s)\,u_{xx}(t+s, x)\,ds\,dx.
\end{aligned}$$

Call the last term $-\Gamma(t)$ and obtain

$$\begin{aligned}
V_2'(t) &= -2\int_0^1 u_{xx}^2\,dx - 2\lambda \int_0^1 u_{xx} f(u)\,dx \\
&\quad - 2\lambda \int_0^1 u_{xx} F(t)\,dx + \Gamma(t) \\
&= -2\int_0^1 u_{xx}^2\,dx - 2\lambda f(u)\,u_x\Big|_0^1 + 2\lambda \int_0^1 f'(u)\,u_x^2\,dx \\
&\quad - 2\lambda \int_0^1 u_{xx} F(t)\,dx + \Gamma(t) \\
&\leqslant -2\int_0^1 u_{xx}^2\,dx - 2\lambda f(0) \int_0^1 u_{xx}\,dx \\
&\quad + 2\lambda \int_0^1 [k + \varphi(u)]\,u_x^2\,dx \\
&\quad - 2\lambda \int_0^1 u_{xx} F(t)\,dx + \Gamma(t) \\
&\leqslant -2(1-\delta) \int_0^1 u_{xx}^2\,dx + 2\lambda k \int_0^1 u_x^2\,dx \\
&\quad + 2\lambda \int_0^1 \varphi(u)\,u_x u_x\,dx + \Gamma(t) + \bar{M}
\end{aligned}$$

(where $\delta>0$ is a small as we please and $\bar{M}>0$). We now integrate the third term by parts and obtain $\int_0^1 u_x \varphi(u)\, u_x\, dx = -\int_0^1 u_{xx} \int_0^u \varphi(s)\, ds\, dx$. Since $|\int_0^{\pm\infty} \varphi(s)\, ds| < \infty$, this last term is bounded by $D \int_0^1 |u_{xx}|\, dx$ for some $D>0$. Therefore, this term can be absorbed into the δ and $\bar{M}$ relations in our last calculation of V_2', yielding

$$V_2'(t) \leqslant -2(1-\delta) \int_0^1 u_{xx}^2\, dx + 2\lambda k \int_0^1 u_x^2\, dx + \Gamma(t) + \bar{M}.$$

Still, $\delta>0$ is as small as we please. Hence

$$\begin{aligned} 0 &= V(T) - V(0) \\ &\leqslant -2(1-\delta) \int_0^T \int_0^1 u_{xx}^2\, dx\, dt + 2\lambda k \int_0^T \int_0^1 u_x^2\, dx\, dt \\ &\quad + \int_0^T \Gamma(t)\, dt + \bar{M}T \\ &\leqslant -2(1-\delta) \int_0^T \int_0^1 u_{xx}^2\, dx + 2kc_1 + \int_0^T \Gamma(t)\, dt + \bar{M}T \end{aligned}$$

using (28). Now

$$\begin{aligned} \int_0^T \Gamma(t)\, dt &= -2\lambda \int_0^T \int_0^1 u_{xx}(t,x) \int_{-\infty}^0 B(t, t+s)\, u_{xx}(t+s, x)\, ds\, dx\, dt \\ &\leqslant \int_0^T \int_0^1 \int_{-\infty}^0 |B(t, t+s)|\, (u_{xx}^2(t,x) + u_{xx}^2(t+s, x))\, ds\, dx\, dt \\ &\leqslant \int_{-\infty}^0 \tilde{B}(s)\, ds \int_0^T \int_0^1 u_{xx}^2\, dx\, dt \\ &\quad + \int_{-\infty}^0 \tilde{B}(s) \int_0^T \int_0^1 u_{xx}^2(t+s, x)\, dx\, dt\, ds \\ &= 2 \int_{-\infty}^0 \tilde{B}(s)\, ds \int_0^T \int_0^1 u_{xx}^2\, dx\, dt. \end{aligned}$$

This yields

$$0 \leqslant -2\left(1 - \delta - \int_{-\infty}^0 \tilde{B}(s)\, ds\right) \int_0^T \int_0^1 u_{xx}^2\, dx + M_1 c_1 + \bar{M}T$$

and so

$$\int_0^T \int_0^1 u_{xx}^2(t,x)\, dx\, dt \leqslant c_2. \tag{29}$$

By (28), there is a $t_1 \in [0, T]$ with

$$T \int_0^1 u_x^2(t_1, x)\, dx \leqslant c_1$$

so that $V_2(t_1) \leqslant c_1/T$. Then for $t_1 \leqslant t \leqslant t_1 + T$ we have

$$\begin{aligned} V_2(t) &\leqslant V_2(t_1) + 2\lambda k \int_{t_1}^{t_1+T} \int_0^1 u_x^2\, dx\, dt \\ &\quad + \int_{t_1}^{t_1+T} |\Gamma(t)|\, dt + \bar{M}T \\ &\leqslant V_2(t_1) + P \qquad \text{for} \quad t \in [t_1, t_1 + T], \text{ some } P > 0. \end{aligned}$$

But

$$V_2(t) = \int_0^1 u_x^2\, dx$$

and

$$\pi^2 \int_0^1 u^2\, dx \leqslant \int_0^1 u_x^2\, dx$$

so

$$\sup_{t_1 \leqslant t \leqslant t_1 + T} \int_0^1 [u_x^2(t, x) + u^2(t, x)]\, dx \leqslant 2(P + (c_1/T)).$$

Thus

$$\sup_{t \in [0, T]} |u(t)|_{H^1} < B_1, \qquad \text{some} \quad B_1 > 0.$$

This completes the proof of Lemma 7.

Suppose that

$f \in C^1$, F is Lipschitz. Thus, we write
(a) for each $D > 0$, there is a $K > 0$ such that $|t|, |s| \leqslant D$ implies that $|f(t) - f(s)| \leqslant K\, |t - s|$, (30)
(b) there is a $K_1 > 0$ such that $|F(t) - F(s)| \leqslant K_1\, |t - s|$.

Let $R(t, s)$ be the unique solution of

$$R(t, s) + \lambda \int_s^t B(t, u)\, R(u, s)\, du = 1, \qquad R(s, s) = 1. \tag{31}$$

We note that there is an $\bar{M} > 0$ with

$$\begin{aligned} &\text{(a)} \quad \int_{-\infty}^t |R_s(t, s)|\, ds \leqslant \bar{M} \qquad \text{for all } t, \\ &\text{(b)} \quad R(t+T, s+T) = R(t, s). \end{aligned} \tag{32}$$

To see (a), we have from (31) that

$$R_s(t, s) - \lambda B(t, s)\, R(s, s) + \lambda \int_s^t B(t, u)\, R_s(u, s)\, du = 0.$$

Thus,

$$|R_s(t, s)| \leqslant |B(t, s)| + \int_s^t |B(t, u)|\, |R_s(u, s)|\, du.$$

For any $a < t$ we have

$$\begin{aligned} \int_a^t |R_s(t, s)|\, ds \leqslant & \int_a^t |B(t, s)|\, ds \\ & + \int_a^t \int_s^t |B(t, u)|\, |R_s(u, s)|\, du\, ds \end{aligned}$$

so that

$$\begin{aligned} \int_a^t |R_s(t, s)|\, ds \leqslant & \int_{-\infty}^t |B(t, s)|\, ds \\ & + \int_a^t |B(t, u)| \int_a^u |R_s(u, s)|\, ds\, du. \end{aligned}$$

Let $y(t) = \int_a^t |R_s(t, s)|\, ds$; then

$$y(t) \leqslant \int_a^t |B(t, u)|\, y(u)\, du + \int_{-\infty}^0 \tilde{B}(v)\, dv, \qquad y(a) = 0.$$

Consider $t \geqslant a$ for which $y(s) \leqslant y(t)$ for $a \leqslant s \leqslant t$. Then

$$y(t) \leqslant y(t) \int_{-\infty}^t |B(t, u)|\, du + \int_{-\infty}^0 \tilde{B}(v)\, dv$$

so that

$$y(t) \leqslant \int_{-\infty}^{0} \tilde{B}(v)\,dv \Big/ \left(1 - \int_{-\infty}^{0} \tilde{B}(v)\,dv\right) =: \bar{M}.$$

Letting $a \to -\infty$ and recalling that $\int_{-\infty}^{0} \tilde{B}(v)\,dv < 1$ we obtain $\int_{-\infty}^{t} |R_s(t, s)|\,ds \leqslant \bar{M}$, as claimed. Now we prove (b). We have

$$R(t+T, s+T) + \lambda \int_{s+T}^{T+t} B(t+T, u)\,R(u, s+T)\,du = 1$$

and

$$R(t+T, s+T) + \lambda \int_{s}^{t} B(t, v)\,R(T+v, T+s)\,dv = 1.$$

Note that $R(T+t, T+t) = 1$, so $R(t+T, s+T)$ is also the solution of (31). By uniqueness we have $R(t+T, s+T) = R(t, s)$. This proves (b).

Let $u(t) \in \tilde{W}$ be given and consider the integral equation in $\tilde{W}$

$$\varphi(t) + \lambda \int_{-\infty}^{t} B(t, s)\,\varphi(s)\,ds = u(t). \tag{33}$$

LEMMA 8. *Suppose that* (24), (25) *hold. Then*

$$\tilde{u}(t) = u(t) - \int_{-\infty}^{t} R_s(t, s)\,u(s)\,ds \tag{34}$$

is a T-periodic solution of (33), $\tilde{u} \in \tilde{W}$.

Proof. We need only show that (34) satisfies (33). Since $u(t) \in \tilde{W}$ and (32)(b) holds, then $\int_{-\infty}^{t} R_s(t, s)\,u(s)\,ds \in \tilde{W}$. Now

$$\begin{aligned} &\tilde{u}(t) + \lambda \int_{-\infty}^{t} B(t, s)\,\tilde{u}(s)\,ds \\ &\quad = u(t) - \int_{-\infty}^{t} R_s(t, s)\,u(s)\,ds + \lambda \int_{-\infty}^{t} B(t, s)\,\tilde{u}(s)\,ds. \end{aligned}$$

Next, we show that

$$-\int_{-\infty}^{t} R_s(t, s)\,u(s)\,ds + \lambda \int_{-\infty}^{t} B(t, s)\,\tilde{u}(s)\,ds = 0.$$

We have

$$R_s(t, s) - \lambda B(t, s) + \lambda \int_{s}^{t} B(t, u)\,R_s(u, s)\,du = 0$$

and

$$-\int_{-\infty}^{t} R_s(t, s)\, u(s)\, ds$$
$$= \int_{-\infty}^{t} u(s) \left[-\lambda B(t, s) + \int_{s}^{t} \lambda B(t, u)\, R_s(u, s)\, du \right] ds$$
$$= -\lambda \int_{-\infty}^{t} B(t, v) \left[u(v) - \int_{-\infty}^{v} R_s(v, s)\, u(s)\, ds \right] dv.$$

Thus,

$$-\int_{-\infty}^{t} R_s(t, s)\, u(s)\, ds = -\lambda \int_{-\infty}^{t} B(t, v)\, \tilde{u}(v)\, dv.$$

This completes the proof.

Consider the following equation in $\tilde{W}$

$$u_t = u_{xx} + \lambda \left[f(\tilde{u}) + F(t) + B(t, t)\, \tilde{u}(t) + \int_{-\infty}^{t} B_t(t, s)\, \tilde{u}(s)\, ds \right], \tag{35}$$

where $u(t, 0) = u(t, 1) = 0$ and

$$\tilde{u}(t) = u(t) - \int_{-\infty}^{t} R_s(t, s)\, u(s)\, ds.$$

LEMMA 9. *If $u(t)$ is a T-periodic solution of* (35), *then* $|u|_{\tilde{W}} \leqslant (1 + \bar{M})\, B_1$ (*where $\bar{M}$ is defined in* (32) *and B_1 is given in Lemma* 7) *and* (22) *has a T-periodic solution.*

Proof. Since $\tilde{u}$ satisfies $\tilde{u}(t) = u(t) - \int_{-\infty}^{t} R_s(t, s)\, u(s)\, ds$, according to Lemma 8 we have that $\tilde{u}(t)$ is a T-periodic solution of (33):

$$\tilde{u}(t) + \lambda \int_{-\infty}^{t} B(t, s)\, \tilde{u}(s)\, ds = u(t).$$

Now we show that $\tilde{u}(t)$ is a T-periodic solution of (22). In fact,

$$u_t = \tilde{u}_t(t) + \lambda B(t, t)\, \tilde{u}(t) + \lambda \int_{-\infty}^{t} B_t(t, s)\, \tilde{u}(s)\, ds \tag{36}$$

and

$$u_{xx} = \tilde{u}_{xx} + \lambda \int_{-\infty}^{t} B(t, s)\, \tilde{u}_{xx}(s)\, ds \tag{37}$$

(since $\tilde{u}(t)=u(t)-\int_{-\infty}^{t} R_s(t,s)\,u(s)\,ds\in D(A)$). Now, substitute (36), (37) into (35) to obtain

$$\tilde{u}_t=\tilde{u}_{xx}+\lambda\left[f(\tilde{u})+F(t)+\int_{-\infty}^{t} B(t,s)\,\tilde{u}_{xx}(s)\,ds\right].$$

Thus, $\tilde{u}$ is a T-periodic solution of (22). By Lemma 7, it follows that

$$\begin{aligned}|u|_{\tilde{W}}&\leqslant|\tilde{u}|_{\tilde{W}}+\int_{-\infty}^{t}|B(t,s)|\,ds\,|\tilde{u}|_{\tilde{W}}\\&\leqslant\left(1+\int_{-\infty}^{0}\tilde{B}(v)\,dv\right)B_1.\end{aligned}$$

This proves Lemma 9.

Let

$$C(t,u(\cdot))=f(\tilde{u})+F(t)+B(t,t)\,\tilde{u}(t)+\int_{-\infty}^{t} B_t(t,s)\,\tilde{u}(s)\,ds \qquad (38)$$

with $\tilde{u}$ defined in (34).

LEMMA 10. *Suppose* (24), (25), (30) *hold. Then* $C(t,\varphi(\cdot))$ *given in* (38) *is Hölder continuous for* $\varphi\in\tilde{W}$ *and* φ *Hölder continuous.*

Proof. Let $\varphi\in\tilde{W}$ be Hölder continuous. Thus, there is a $k\in(0,1]$ and an $L_1>0$ such that

$$|\varphi(t_1)-\varphi(t_2)|_{H^0}\leqslant L_1\,|t_1-t_2|^k.$$

For $t_1>t_2$ and $t_1,t_2\in[0,T]$, we have

$$\begin{aligned}|\tilde{\varphi}(t_1)-\tilde{\varphi}(t_2)|_{H^0}&=\left|\varphi(t_1)-\varphi(t_2)-\left(\int_{-\infty}^{t_1} R_s(t_1,s)\,\varphi(s)\right.\right.\\&\qquad\left.\left.-\int_{-\infty}^{t_2} R_s(t_2,s)\,\varphi(s)\,ds\right)\right|_{H^0}\\&\leqslant|\varphi(t_1)-\varphi(t_2)|_{H^0}+\left|\int_{-\infty}^{t_1} R_s(t_1,s)\,\varphi(s)\,ds\right.\\&\qquad\left.-\int_{-\infty}^{t_2} R_s(t_2,s)\,\varphi(s)\,ds\right|_{H^0}\end{aligned}$$

$$\leqslant |\varphi(t_1)-\varphi(t_2)|_{H^0}+\left|\int_{t_2}^{t_1} R_s(t_1, s)\,\varphi(s)\,ds\right|_{H^0}$$

$$+\left|\int_{-\infty}^{t_2} (R_s(t_1, s)-R_s(t_2, s))\,\varphi(s)\,ds\right|_{H^0}$$

$$\leqslant |\varphi(t_1)-\varphi(t_2)|_{H^0}+R^*\,|\varphi|_{\tilde{W}}\,|t_1-t_2|$$

$$+\int_{-\infty}^{t_2} |R_s(t_1, s)-R_s(t_2, s)|\,ds\,|\varphi|_{\tilde{W}}, \tag{39}$$

where $R^*=\sup\{|R_s(t, s)|: (t, s)\in[0, T]\times[0, T]\}$. For $s\leqslant t_1$, we have

$$R_s(t_1, s)-R_s(t_2, s)=B(t_1, s)-B(t_2, s)$$

$$+\int_s^{t_1} (B(t_1, v)-B(t_2, v))\,R_s(v, s)\,dv$$

$$+\int_{t_2}^{t_1} B(t_2, v)\,R_s(v, s)\,dv$$

so that

$$|R_s(t_1, s)-R_s(t_2, s)|\leqslant |B(t_1, s)-B(t_2, s)|$$

$$+\int_s^{t_1} |B(t_1, v)-B(t_2, v)|\,|R_s(v, s)|\,dv$$

$$+\int_{t_2}^{t_1} |B(t_2, v)|\,|R_s(v, s)|\,dv$$

$$\leqslant |t_1-t_2|^{\theta}\,b(s)+|t_1-t_2|^{\theta}\int_s^{t_1} b(v)\,|R_s(v, s)|\,dv$$

$$+B^*\int_{t_2}^{t_1} |R_s(v, s)|\,dv,$$

where $B^*=\sup\{|B(t, s)|: (t, s)\in[0, T]\times[0, T]\}$. Then

$$\int_{-\infty}^{t_2} |R_s(t_1, s)-R_s(t_2, s)|\,ds$$

$$\leqslant |t_1-t_2|^{\theta}\int_{-\infty}^{t_2} b(v)\,dv$$

$$+|t_1-t_2|^{\theta}\int_{-\infty}^{t_2}\int_s^{t_1} b(v)\,|R_s(v, s)|\,dv\,ds$$

$$+B^*\int_{-\infty}^{t_2}\int_{t_2}^{t_1} |R_s(v, s)|\,dv\,ds$$

$$\leqslant |t_1 - t_2|^\theta \int_{-\infty}^{T} b(v)\, dv$$

$$+ |t_1 - t_2|^\theta \int_{-\infty}^{t_1} b(v) \int_{-\infty}^{v} |R_s(v, s)|\, ds\, dv$$

$$+ B^* \int_{t_2}^{t_1} \int_{-\infty}^{v} |R_s(v, s)|\, ds\, dv$$

$$\leqslant |t_1 - t_2|^\theta \left(\int_{-\infty}^{T} b(v)\, dv \right) (1 + \bar{M}) + B^* \bar{M}\, |t_1 - t_2|$$

(where $\bar{M}$ is defined in (32)) so that there is an $M_2 > 0$ with

$$\int_{-\infty}^{t_2} |R_s(t_1, s) - R_s(t_2, s)|\, ds \leqslant M_2\, |t_1 - t_2|^\theta. \tag{40}$$

Thus,

$$\begin{aligned} &|\tilde{\varphi}(t_1) - \tilde{\varphi}(t_2)|_{H^0} \\ &\quad \leqslant |\varphi(t_1) - \varphi(t_2)|_{H^0} + R^*\, |\varphi|_{\tilde{W}}\, |t_1 - t_2| + M_2\, |\varphi|_{\tilde{W}}\, |t_1 - t_2|^\theta \\ &\quad \leqslant L_1\, |t_1 - t_2|^k + R^*\, |\varphi|_{\tilde{W}}\, |t_1 - t_2| + |\varphi|_{\tilde{W}}\, M_2\, |t_1 - t_2|^\theta \end{aligned}$$

and so

$$|\tilde{\varphi}(t_1) - \tilde{\varphi}(t_2)|_{H^0} \leqslant M_3\, |t_1 - t_2|^\gamma, \tag{41}$$

where $\gamma = \min\{\theta, k\}$ and M_3 depends on $|\varphi|_{\tilde{W}}$.

For $t_1, t_2 \in [0, T]$ we have

$$\begin{aligned} &|B(t_1, t_1)\, \tilde{\varphi}(t_1) - B(t_2, t_2)\, \tilde{\varphi}(t_2)|_{H^0} \\ &\quad \leqslant |B(t_1, t_1)\, \tilde{\varphi}(t_1) - B(t_1, t_1)\, \tilde{\varphi}(t_2)|_{H^0} \\ &\qquad + |B(t_1, t_1)\, \tilde{\varphi}(t_2) - B(t_2, t_2)\, \tilde{\varphi}(t_2)|_{H^0} \\ &\quad \leqslant |B(t_1, t_1)|\, |\tilde{\varphi}(t_1) - \tilde{\varphi}(t_2)|_{H^0} \\ &\qquad + |B(t_1, t_1) - B(t_2, t_2)|\, |\tilde{\varphi}(t_2)|_{H^0}. \end{aligned}$$

From (25), we have

$$\begin{aligned} &|B(t, t)| \leqslant B_1 \text{ for some } B_1 > 0 \text{ and all } t \in [0, T], \\ &|B(t, t) - B(s, s)| \leqslant L_B\, |t - s| \text{ for some } L_B > 0 \\ &\qquad\qquad\qquad\qquad\qquad \text{and all } t, s \in [0, T]. \end{aligned} \tag{42}$$

Thus,

$$|B(t_1, t_1)\,\tilde{\varphi}(t_1) - B(t_2, t_2)\,\tilde{\varphi}(t_2)|_{H^0} \\ \leqslant B_1\,|\tilde{\varphi}(t_1) - \tilde{\varphi}(t_2)|_{H^0} + L_B\,|t_1 - t_2|\,|\tilde{\varphi}(t_2)|_{H^0}.$$

Note that from (41)

$$|\tilde{\varphi}(t_1) - \tilde{\varphi}(t_2)|_{H^0} \leqslant M_3\,|t_1 - t_2|^\gamma,$$

while

$$|\tilde{\varphi}|_{\bar{W}} \leqslant (1 + \bar{M})\,|\varphi|_{\bar{W}},$$

where $\bar{M}$ is defined in (32). Then there is an $M_4 > 0$ depending on $|\varphi|_{\bar{W}}$ such that

$$|B(t_1, t_2)\,\tilde{\varphi}(t_1) - B(t_2, t_2)\,\tilde{\varphi}(t_2)|_{H^0} \leqslant M_4\,|t_1 - t_2|^\gamma. \tag{43}$$

As before, we denote by $(\partial/\partial_1)\,B(t, s)$ the derivative of B with respect to the first argument of B. Then

$$\begin{aligned}
&\left|\int_{-\infty}^{t_1} B_t(t_1, s)\,\tilde{\varphi}(s)\,ds - \int_{-\infty}^{t_2} B_t(t_2, s)\,\tilde{\varphi}(s)\,ds\right|_{H^0} \\
&\quad = \left|\int_{-\infty}^{0} (\partial/\partial_1)\,B(t_1, t_1 + s)\,\tilde{\varphi}(t_1 + s)\,ds\right. \\
&\qquad \left. - \int_{-\infty}^{0} (\partial/\partial_1)\,B(t_2, t_2 + s)\,\tilde{\varphi}(t_2 + s)\,ds\right|_{H^0} \\
&\quad \leqslant \int_{-\infty}^{0} |(\partial/\partial_1)\,B(t_1, t_1 + s)|\,|\tilde{\varphi}(t_1 + s) - \tilde{\varphi}(t_2 + s)|_{H^0}\,ds \\
&\qquad + \int_{-\infty}^{0} |(\partial/\partial_1)\,B(t_1, t_1 + s) \\
&\qquad - (\partial/\partial_1)\,B(t_2, t_2 + s)|\,|\tilde{\varphi}(t_2 + s)|_{H^0}\,ds \\
&\quad \leqslant D_2 M_3\,|t_1 - t_2|^\gamma + |\varphi|_{\bar{W}}\,(1 + \bar{M}) \int_{-\infty}^{0} b^*(v)\,dv\,|t_1 - t_2|^\alpha \\
&\quad \leqslant M_5\,|t_1 - t_2|^{\alpha_1}
\end{aligned} \tag{44}$$

where $\alpha_1 = \min\{\alpha, \gamma\}$ and M_5 is a positive constant depending on $|\varphi|_{\bar{W}}$. In (30) for $D = (1 + \bar{M})\,|\varphi|_{\bar{W}}$, there is a $K > 0$ such that $|f(u) - f(v)| \leqslant K\,|u - v|$ for $|u|, |v| \leqslant D$. Using (41) we have

$$\begin{aligned} |f(\tilde{\varphi}(t_1)) - f(\tilde{\varphi}(t_2))|_{H^0} &\leqslant K\,|\tilde{\varphi}(t_1) - \tilde{\varphi}(t_2)|_{H^0} \\ &\leqslant KM_3\,|t_1 - t_2|^{\gamma} \end{aligned} \tag{45}$$

and

$$|F(t_1) - F(t_2)|_{H^0} \leqslant K_1\,|t_1 - t_2|.$$

Combining (43), (44), (45), there is a $\beta \in (0, 1]$ and a constant K_2 depending on $|\varphi|_{\tilde{W}}$ such that

$$|C(t_1, \varphi(\cdot)) - C(t_2, \varphi(\cdot))|_{H^0} \leqslant K_2\,|t_1 - t_2|^{\beta}.$$

This completes the proof of Lemma 10.

LEMMA 11. *Suppose that* (24), (25), (26), *and* (30) *hold. Then* C: $[0, T] \times \tilde{W} \to H^0$ *takes bounded sets into bounded sets and condition* (iv) *of Theorem* 1 *holds.*

Proof. By (30)(a), for any $\varphi \in \tilde{W}$ there is a $k_3 > 0$ depending on $|\varphi|_{\tilde{W}}$ such that $|f(\varphi)|_{H^0} \leqslant k_3\,|\varphi|_{H^0} + |f(0)|$. Let $\|F\| = \sup\{|F(t)| : 0 \leqslant t \leqslant T\}$, then

$$|C(t, \varphi(\cdot))|_{H^0} \leqslant (k_3 + B_1 + D_2)(1 + \bar{M})\,|\varphi|_{\tilde{W}} + |f(0)| + \|F\|$$

(where B_1 is defined in (42), $\bar{M}$ is given in (32)) for all $t \in [0, T]$.

For $\varphi, \psi \in X$ we have

$$\begin{aligned} &|C(t, \varphi(\cdot)) - C(t, \psi(\cdot))|_{H^0} \\ &\qquad \leqslant (K + B_1 + D_2)(1 + \bar{M}) \sup_{t \in [0, T]} |\varphi(t) - \psi(t)|_{H^0}. \end{aligned}$$

Thus, (iv) of Theorem 1 holds and the proof is complete.

THEOREM 3. *Suppose that* (24), (25), (26), *and* (30) *hold. Then* (22) *has a T-periodic solution for* $\lambda = 1$.

Proof. By Lemmas 9, 10, 11, we know that (35) has a T-periodic solution for $\lambda = 1$. From Lemma 9 it follows that *(22) has a T-Periodic solution* for $\lambda = 1$.

We now show that the methods of establishing boundedness in this paper do not establish dissipativeness; in fact, the conditons can be satisfied and there are still unbounded solutions.

EXAMPLE 3. Consider the scalar equation

$$u_t = -(2/\pi^2) \int_{-\infty}^{t} e^{-(t-s)} u_{xx}(s)\, ds + \lambda F(t) \tag{46}$$

with $u(t, 0) = u(t, 1) = 0$, $F(t+T) = F(t)$ for some $T > 0$. Also, let F be differentiable.

THEOREM 4. *There is an $M > 0$ such that any T-periodic solution of* (46) *satisfies*

$$\sup_{t \in [0, T]} \int_0^1 \{u^2(t, x) + u_x^2(t, x)\}\, dx \leqslant M.$$

Moreover, if (46) *has a T-periodic solution, then it has unbounded solutions.*

LEMMA 12. *Let φ be a continuous T-periodic function and $a > 0$. Then*

$$\int_0^T \varphi(t) \int_{-\infty}^{t} e^{-a(t-s)} \varphi(s)\, ds\, dt = a \int_0^T \left[\int_{-\infty}^{t} e^{-a(t-s)} \varphi(s)\, ds \right]^2 dt.$$

Proof. Define

$$V(t) = \int_{-\infty}^{t} e^{-a(t-s)} \varphi(s)\, ds$$

so that

$$V'(t) = \varphi(t) - a \int_{-\infty}^{t} e^{-a(t-s)} \varphi(s)\, ds = \varphi(t) - aV(t)$$

and

$$V(t)\, V'(t) = \varphi(t)\, V(t) - aV^2(t).$$

Thus,

$$\int_0^T V(t)\, V'(t)\, dt = \int_0^T \varphi(t)\, V(t)\, dt - a \int_0^T V^2(t)\, dt$$

so that

$$\frac{1}{2}(V^2(T) - V^2(0)) = \int_0^T \varphi(t)\, V(t)\, dt - a \int_0^T V^2(t)\, dt.$$

Since $V(t)$ is a T-periodic function, we have $V(T)=V(0)$ so that

$$\int_0^T \varphi(t)\, V(t)\, dt = a\int_0^T V^2(t)\, dt = a\int_0^T \left(\int_{-\infty}^t e^{-a(t-s)}\varphi(s)\, ds\right)^2 dt,$$

as required.

Now, to prove Theorem 4 define

$$V_1(t)=\frac{1}{2}\int_0^1 u_x^2(t,x)\, dx,$$

$$V_2(t)=\lambda F(t)\int_0^1\int_{-\infty}^t e^{-(t-s)}u_{xx}(s,x)\, ds\, dx,$$

and

$$V_3(t)=V_1(t)+V_2(t).$$

Then

$$\begin{aligned} V_1'(t) &= \int_0^1 u_x u_{xt}\, dx = -\int_0^1 u_t u_{xx}\, dx \\ &= (2/\pi^2)\int_0^1 u_{xx}(t,x)\int_{-\infty}^t e^{-(t-s)}u_{xx}(s,x)\, ds\, dx \\ &\quad -\lambda F(t)\int_0^1 u_{xx}(t,x)\, dx, \end{aligned} \tag{47}$$

$$\begin{aligned} V_2'(t) &= \lambda F'(t)\int_0^1\int_{-\infty}^t e^{-(t-s)}u_{xx}(s,x)\, ds\, dx \\ &\quad +\lambda F(t)\int_0^1 u_{xx}(t,x)\, dx \\ &\quad -\lambda F(t)\int_0^1\int_{-\infty}^t e^{-(t-s)}u_{xx}(s,x)\, ds\, dx \end{aligned} \tag{48}$$

and

$$\begin{aligned} V_3'(t) &= (2/\pi^2)\int_0^1 u_{xx}(t,x)\int_{-\infty}^t e^{-(t-s)}u_{xx}(s,x)\, ds\, dx \\ &\quad +\lambda F'(t)\int_0^1\int_{-\infty}^t e^{-(t-s)}u_{xx}(s,x)\, ds\, dx \\ &\quad -\lambda F(t)\int_0^1\int_{-\infty}^t e^{-(t-s)}u_{xx}(s,x)\, ds\, dx. \end{aligned} \tag{49}$$

Use the fact that V_3 is T-periodic and integrate (49) to obtain

$$\begin{aligned}
0 &= V_3(T) - V_3(0) \\
&= (2/\pi^2) \int_0^T \int_0^1 u_{xx}(t, x) \int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds\, dx\, dt \\
&\quad + \lambda \int_0^T \int_0^1 F'(t) \int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds\, dx\, dt \\
&\quad - \lambda \int_0^T \int_0^1 F(t) \int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds\, dx\, dt.
\end{aligned}$$

By Lemma 12 it follows that

$$\begin{aligned}
&(2/\pi^2) \int_0^1 \int_0^T \left[\int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds \right]^2 dt\, dx \\
&\quad = (2/\pi^2) \int_0^1 \int_0^T u_{xx}(t, x) \int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds\, dt\, dx \\
&\quad \leqslant \int_0^T \int_0^1 |F'(t)| \left| \int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds \right| dx\, dt \\
&\qquad + \int_0^T \int_0^1 |F(t)| \left| \int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds \right| dx\, dt.
\end{aligned}$$

Thus, there is a $C_1 > 0$ such that

$$\int_0^1 \int_0^T \left[\int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds \right]^2 dt\, dx \leqslant C_1. \tag{50}$$

From (48) we have

$$\begin{aligned}
\lambda F(t) \int_0^1 u_{xx}(t, x)\, dx &= V_2'(t) \\
&\quad + \lambda F(t) \int_0^1 \int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds\, dx \\
&\quad - \lambda F'(t) \int_0^1 \int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds\, dx.
\end{aligned}$$

Choose $t_1 \in [0, T]$ such that

$$V_2(t_1) = \max\{ V_2(t) \colon t \in [t_1, t_1 + T] \}.$$

Then for each $t \in [t_1, t_1 + T]$ we have

$$\begin{aligned}
\lambda \int_{t_1}^{t} F(\tau) \int_0^1 u_{xx}(\tau, x)\, dx\, d\tau &= \int_{t_1}^{t} V_2'(\tau)\, d\tau \\
&\quad + \lambda \int_{t_1}^{t} F(\tau) \int_0^1 \int_{-\infty}^{\tau} e^{-(\tau-s)} u_{xx}(s, x)\, ds\, dx\, d\tau \\
&\quad - \lambda \int_{t_1}^{t} F'(\tau) \int_0^1 \int_{-\infty}^{\tau} e^{-(\tau-s)} u_{xx}(s, x)\, ds\, dx\, d\tau \\
&\leqslant \int_{t_1}^{t_1+T} |F(\tau)| \int_0^1 \left| \int_{-\infty}^{\tau} e^{-(\tau-s)} u_{xx}(s, x)\, ds \right| dx\, d\tau \\
&\quad + \int_{t_1}^{t_1+T} |F'(\tau)| \int_0^1 \left| \int_{-\infty}^{\tau} e^{-(\tau-s)} u_{xx}(s, x)\, ds \right| dx\, d\tau \\
&= \int_0^T (|F(\tau)| + |F'(\tau)|) \int_0^1 \left| \int_{-\infty}^{\tau} e^{-(\tau-s)} u_{xx}(s, x)\, ds \right| dx\, d\tau.
\end{aligned}$$

Using (50) we see that for each $t \in [t_1, t_1 + T]$ we have

$$\lambda \int_{t_1}^{t} F(\tau) \int_0^1 u_{xx}(\tau, x)\, dx\, d\tau \leqslant C_2, \qquad \text{some} \quad C_2 > 0.$$

Thus,

$$\lambda \int_0^{t} F(\tau) \int_0^1 u_{xx}(\tau, x)\, dx\, d\tau \leqslant C_2 \qquad \text{for all} \quad t \in [0, T]. \tag{51}$$

From (47) we have

$$\begin{aligned}
(2/\pi^2) \int_0^1 u_{xx}(t, x) \int_{-\infty}^{t} e^{-(t-s)} u_{xx}(s, x)\, ds\, dx \\
= V_1'(t) + \gamma F(t) \int_0^1 u_{xx}(t, x)\, dx.
\end{aligned}$$

Find $t_2 \in [0, T]$ such that

$$V_1(t_2) = \max\{V_1(t) : t \in [t_2, t_2 + T]\}.$$

Then for all $t \in [t_2, t_2 + T]$ we have

$$\begin{aligned}
(2/\pi^2) \int_{t_2}^{t} \int_0^1 u_{xx}(\tau, x) \int_{-\infty}^{\tau} e^{-(\tau-s)} u_{xx}(s, x)\, ds\, dx\, d\tau \\
= \int_{t_1}^{t} V_1'(\tau)\, d\tau + \lambda \int_{t_1}^{t} F(\tau) \int_0^1 u_{xx}(\tau, x)\, dx\, d\tau \\
\leqslant \lambda \int_{t_1}^{t} F(\tau) \int_0^1 u_{xx}(\tau, x)\, dx\, d\tau \leqslant C_2
\end{aligned}$$

by (51). Thus, for all $t \in [0, T]$ we have

$$(2/\pi^2) \int_0^t \int_0^1 u_{xx}(\tau, x) \int_{-\infty}^{\tau} e^{-(\tau - s)} u_{xx}(s, x)\, ds\, dx\, d\tau \leqslant C_2. \tag{52}$$

From (48) we write

$$\begin{aligned} -\lambda F(t) \int_0^1 u_{xx}(t, x)\, dx \\ &= -V_2'(t) + \lambda F'(t) \int_0^1 \int_{-\infty}^{t} e^{-(t-s)} u_{xx}(s, x)\, ds\, dx \\ &\quad - \lambda F(t) \int_0^1 \int_{-\infty}^{t} e^{-(t-s)} u_{xx}(s, x)\, ds\, dx \end{aligned}$$

and find $t_3 \in [0, T]$ such that

$$V_2(t_3) = \min\{V_2(t) \colon t \in [t_3, t_3 + T]\}.$$

Then for all $t \in [t_3, t_3 + T]$ we have

$$\begin{aligned} -\lambda \int_{t_3}^t F(\tau) \int_0^1 u_{xx}(\tau, x)\, dx\, d\tau &= -\int_{t_3}^t V_2'(\tau)\, d\tau \\ &\quad + \lambda \int_{t_3}^t F'(\tau) \int_0^1 \int_{-\infty}^{\tau} e^{-(\tau - s)} u_{xx}(s, x)\, ds\, dx\, d\tau \\ &\quad - \lambda \int_{t_3}^t F(\tau) \int_0^1 \int_{-\infty}^{\tau} e^{-(\tau - s)} u_{xx}(s, x)\, ds\, dx\, d\tau \\ &\leqslant \int_{t_3}^{t_3 + T} (|F(\tau)| + |F'(\tau)|) \int_0^1 \left| \int_{-\infty}^{\tau} e^{-(\tau - s)} u_{xx}(s, x)\, ds \right| dx\, d\tau. \end{aligned}$$

Using (50) and boundedness of F, F' we obtain

$$-\lambda \int_{t_3}^t F(\tau) \int_0^1 u_{xx}(\tau, x)\, dx\, d\tau \leqslant C_3 \qquad \text{for some} \quad C_3 > 0.$$

Thus,

$$-\lambda \int_0^t F(\tau) \int_0^1 u_{xx}(\tau, x)\, dx\, d\tau \leqslant C_3. \tag{53}$$

Differentiating (46), we have

$$u_{tt} + (2/\pi^2)\, u_{xx} = (2/\pi^2) \int_{-\infty}^{t} e^{-(t-s)} u_{xx}(s, x)\, ds + \lambda F'(t).$$

Multiplying by $u(t, x)$ and integrating from 0 to 1 with respect to x and from 0 to T with respect to t we obtain

$$\begin{aligned}\int_0^T \int_0^1 uu_{tt}\, dx\, dt &+ (2/\pi^2) \int_0^T \int_0^1 uu_{xx}\, dx\, dt \\ &= (2/\pi^2) \int_0^T \int_0^1 u(t, x) \int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds\, dx\, dt \\ &+ \int_0^T \int_0^1 \lambda F'(t)\, u\, dx\, dt. \end{aligned} \tag{54}$$

Note that

$$\int_0^T u_{tt} u\, dt = uu_t \Big|_0^T - \int_0^T u_t^2\, dt = -\int_0^T u_t^2\, dt,$$

$$\int_0^1 uu_{xx}\, dx = uu_x \Big|_0^1 - \int_0^1 u_x^2\, dx = -\int_0^1 u_x^2\, dx,$$

and that

$$\begin{aligned}&\left| 2 \int_0^T \int_0^1 u(t, x) \int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds\, dx\, dt \right| \\ &\quad \leqslant \int_0^T \int_0^1 u^2(t, x)\, dx\, dt \\ &\quad + \int_0^T \int_0^1 \left[\int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds \right]^2 dx\, dt.\end{aligned}$$

Thus, by (54) we get

$$\begin{aligned}&\int_0^T \int_0^1 [u_t^2 + (2/\pi^2)\, u_x^2]\, dx\, dt \\ &\quad \leqslant (1/\pi^2) \int_0^T \int_0^1 u^2(t, x)\, dx\, dt \\ &\quad + (1/\pi^2) \int_0^1 \int_0^T \left[\int_{-\infty}^t e^{-(t-s)} u_{xx}(s, x)\, ds \right]^2 dt\, dx \\ &\quad + \int_0^T \int_0^1 |F'(t)|\, |u|\, dx\, dt.\end{aligned}$$

Using (50), $\pi^2 \int_0^1 u^2\,dx \leqslant \int_0^1 u_x^2\,dx$, and boundedness of $F'(t)$, we have

$$\int_0^T \int_0^1 [u_t^2 + u_x^2]\,dx\,dt \leqslant C_0, \qquad \text{some} \quad C_0 > 0.$$

By the mean value theorem, there is a $t_0 \in [0, T]$ such that

$$\int_0^1 u_x^2(t_0, x)\,dx \leqslant C_0/T.$$

This implies that $V_1(t_0) \leqslant C_0/T$. Integrating (47) and using (52), (53), we get

$$\begin{aligned} V_1(t) &= V_1(t_0) + \int_{t_0}^t V_1'(\tau)\,d\tau \\ &\leqslant (C_0/T) \\ &\quad + (2/\pi^2) \int_{t_0}^t \int_0^1 u_{xx}(\tau, x) \int_{-\infty}^{\tau} e^{-(\tau - s)} u_{xx}(s, x)\,ds\,dx\,d\tau \\ &\quad - \lambda \int_{t_0}^t F(\tau) \int_0^1 u_{xx}(\tau, x)\,dx\,d\tau \\ &\leqslant (C_0/T) + C_2 + C_3. \end{aligned}$$

Hence, there is an $M > 0$ with

$$\sup_{t \in [0, T]} \int_0^1 [u^2(t, x) + u_x^2(t, x)]\,dx \leqslant M,$$

proving the first part of Theorem 4.

Now, let $\varphi_\lambda(t, x)$ be a T-periodic solution of (46). Consider $u(t, x) = (\sin \pi x)\, e^t + \varphi_\lambda(t, x)$. Clearly, $u(t, 0) = u(t, 1)$. Let $\psi(t, x) = (\sin \pi x)\, e^t$ so that

$$\psi_t = (\sin \pi x)\, e^t$$

and so

$$\begin{aligned} -(2/\pi^2) \int_{-\infty}^t e^{-(t-s)} \psi_{xx}(s, x)\,ds &= 2 \sin \pi x \int_{-\infty}^t e^{-t+2s}\,ds \\ &= (\sin \pi x)\, e^t. \end{aligned}$$

Thus,

$$\psi_t = -(2/\pi^2) \int_{-\infty}^{t} e^{-(t-s)} \psi_{xx}(s, x)\, ds.$$

Therefore, $u(t, x)$ is a solution of (1), but unbounded.

Finally, note that if $F(t) = 4/\pi^2$, then $\varphi_\lambda(t, x) = \lambda(x^2 - x)$ is a periodic solution of (46).

Remark. When $B(t, s) = F(t) = 0$, Eq. (11) has been the subject of extensive investigation using a slightly different Liapunov function. For discussion and history see Hale [10, pp. 75–78] (Hale's equation is $u_t = (a(x)\, u_x)_x + f(u)$ which can be treated in the same way), Henry [11, pp. 4, 61, 85, 93, especially 118–125], and Walker [14]. In those discussions it is generally required that f be at least C^1 and that

$$\overline{\lim_{|u| \to \infty}}\ f(u)/u \leqslant 0. \tag{55}$$

The main use of (55) is in showing that solutions are bounded. The goal is to show that all solutions tend to 0. However, if we use the $V = V_1 + V_2$ of Lemma 4, taking $B(t) = F(t) = 0$, we easily get boundedness of solutions under the condition

$$uf(u) \leqslant ku^2 + M$$

for some $M > 0$ and $k \in (0, \pi^2)$. By contrast, (55) asks that for each $\varepsilon > 0$ there exist $K > 0$ such that $|u| \geqslant K$ implies that $f(u)/u \leqslant \varepsilon$ or that $uf(u) \leqslant \varepsilon u^2$. But there is an $\bar{M} > 0$ with $uf(u) \leqslant \bar{M}$ for $|u| \leqslant K$; thus (55) asks that $uf(u) \leqslant \varepsilon u^2 + \bar{M}$ for arbitrarily small ε. Thus, our work extends the classical results even when $B = F = 0$.

References

1. T. A. Burton, P. W. Eloe, and M. N. Islam, Periodic solutions of linear integrodifferential equations, *Math. Nachr.* **147** (1990), 175–184.
2. T. A. Burton, P. W. Eloe, and M. N. Islam, Nonlinear integrodifferential equations and *a priori* bounds on periodic solutions, *Ann. Mat. Pura Appl.*, in press.
3. G. DaPrato, Maximal regularity for abstract differential equations and applications to the existence of periodic solutions, *in* "Trends in the Theory and Practice of Nonlinear Analysis" (V. Lakshmikantham, Ed.), pp. 121–126, North-Holland, Amsterdam, 1985.
4. G. DaPrato and A. Lunardi, Periodic solutions of linear integro-differential equations with infinite delay in Banach spaces, *in* "Differential Equations in Banach Spaces" (A. Favini and E. Obrecht, Eds.), pp. 49–60, Springer-Verlag, New York, 1986.
5. P. W. Eloe and J. Henderson, Nonlinear boundary value problems and *a priori* bounds on solutions, *SIAM J. Math. Anal.* **15** (1984), 642–647.

6. A. FRIEDMAN, "Partial Differential Equations," Krieger Pub. Co., Huntington, New York, 1976.
7. A. GRANAS, Sur la méthode de continuité de Poincaré, *C. R. Acad. Sci. Paris Sér. I Math.* **282** (1976), 983–985.
8. A. GRANAS, R. B. GUENTHER, AND J. W. LEE, Nonlinear boundary value problems for some classes of ordinary differential equations, *Rocky Mountain J. Math.* **10** (1980), 35–58.
9. A. GRANAS, R. B. GUENTHER, AND J. W. LEE, "Nonlinear Boundary Value Problems for Ordinary Differential Equations," Dissertation Mathematicae, Vol. 244, Polish Scientific Publications, Warsaw, 1985.
10. J. K. HALE, "Asymptotic Behavior of Dissipative Systems", Amer. Math. Soc,. Providence, RI, 1988.
11. D. HENRY, "Geometric Theory of Semilinear Parabolic Equations," Springer-Verlag, New York, 1981.
12. A. LUNARDI, Stability of the periodic solutions to fully nonlinear parabolic equations in Banach spaces, *Differential Integral Equations* **1** (1988), 253–279.
13. A. PAZY, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.
14. J. A. WALKER, "Dynamical Systems and Evolution Equations," Plenum, New York, 1980.